The fundamental group

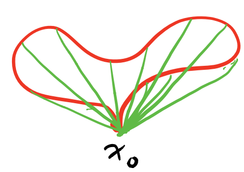
The collection of paths in a space X isn't a group because we can't multiply every pair of objects and there's no identity. In terms of category theory, this is because we have more than one object. If instead we restrict our category to one object (i.e. one point at which all paths begin and end), we get a group.

Def: X a space, $x_0 \in X$ a base point. A path in X from x_0 to itself is a loop. The set of path homotopy classes of loops based at x_0 w/ operation $*$ (concatenation) is called the fundamental group of X, denoted $\pi_1(X, x_0)$.

In terms of the category of paths, this is just $\mathrm{Mor}(x_0, x_0) = \mathrm{Aut}(x_0)$.

Ex: In $\mathbb{R}^n$, every loop f at x_0 is path-homotopic to the identity (i.e. the constant path) by the straight line homotopy:

$$F(s,t) = f(s)(1-t) + t\,x_0.$$



So $\pi_1(\mathbb{R}^n, x_0)$ is trivial.

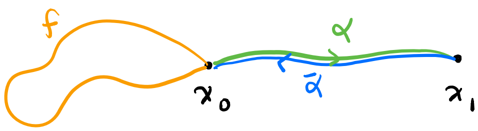
Moreover, any convex subspace of $\mathbb{R}^n$ has trivial fundamental group.

Def: X is called simply-connected if it's nonempty, path-connected, and for $x_0 \in X$, $\pi_1(X, x_0)$ is trivial. (e.g. S^n, $n > 1$)

Dependence on base point

Let $x_0, x_1 \in X$ be in the same path-component of X, and α a path from x_0 to x_1.

Then for any loop f at x_0, we get a loop at x_0 as follows:



So we get a group homomorphism

$$\hat{\alpha} : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_0) \quad \text{given by}$$

$$\hat{\alpha}([f]) = \overset{[\alpha]^{-1}}{\overset{\shortparallel}{[\bar{\alpha}]}} * [f] * [\alpha]$$, which is well-defined since $*$ is well-defined on homotopy classes.

Claim: $\hat{\alpha}$ is a group isomorphism.

Pf: If $A, B \in \pi_1(X, x_0)$, then

$$\begin{aligned} \hat{\alpha}(A * B) &= [\alpha]^{-1} A B [\alpha] \\ &= [\alpha]^{-1} A [\alpha][\alpha]^{-1} B [\alpha] \\ &= \hat{\alpha}(A) * \hat{\alpha}(B). \end{aligned}$$

If $\beta = \bar{\alpha}$, then $\hat{\beta} : \pi_1(X, x_1) \longrightarrow \pi_1(X, x_0)$ and

$$\hat{\alpha}(\hat{\beta}(A)) = [\alpha]^{-1} \underset{\underset{[\alpha]}{\overset{\shortparallel}{\underset{\shortparallel}{[\bar{\alpha}]^{-1}}}}}{[\beta]^{-1}} A \underset{\underset{[\bar{\alpha}]}{\shortparallel}}{[\beta]} [\alpha] = A.$$

(By symmetry, $\hat{\beta} \circ \hat{\alpha} = id$ as well) so $\hat{\alpha}$ is a group isomorphism. $\square$

In particular, if X is path-connected, $\pi_1(X, x_0)$ is independent of x_0 up to isom.

Cor: Any loop f at x_0 induces an automorphism $\hat{f} \in Aut(\pi_1(X, x_0))$ called an <u>inner automorphism</u>.

π_1 as a functor

We want to check that π_1 is a functor from the category of "pointed topological spaces" to the category of groups.

A <u>pointed topological space</u> is a topological space along w/ choice of a basepoint, (X, x_0). In this category, the morphisms are continuous maps that preserve basepoints. i.e.

$$f : (X, x_0) \to (Y, y_0) \text{ s.t. } f: X \to Y \text{ is continuous and } f(x_0) = y_0.$$

Def: Let $h : (X, x_0) \to (Y, y_0)$ be continuous. Define

$$h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0) \text{ by}$$

$$h_*([f]) = [h \circ f]$$

$$\begin{tikzcd} I \arrow[r, "f"] \arrow[d, "h \circ f"'] & X \arrow[dl, "h"] \\ Y & \end{tikzcd}$$

This is the <u>group homomorphism</u> <u>induced by h</u>.

We need to check h_* is well-defined:

if F is a path homotopy between f and g, then $h \circ F$ is a path homotopy between $h \circ f$ and $h \circ g$.

Why is h_* a homomorphism? $(h \circ f) * (h \circ g) = h \circ (f * g)$
h preserves concatenation.

Claim: π_1 is a functor, w/ induced morphisms $\pi_1(h) = h_*$.

Pf: Let $h : (X, x_0) \to (Y, y_0)$ and $k : (Y, y_0) \to (Z, z_0)$ be continuous.
If $[f] \in \pi_1(X, x_0)$, then

$$(k \circ h)_*([f]) = [(k \circ h) \circ f] = [k \circ (h \circ f)] = k_*([h \circ f]) = k_*(h_*([f])).$$

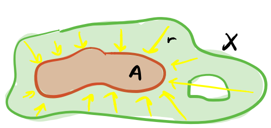
Similarly, $(id_X)_*([f]) = [id_X \circ f] = [f] = id_{\pi_1(X, x_0)}([f])$. $\square$

Cor: If $h : (X, x_0) \to (Y, y_0)$ is a homeomorphism, then
h_* is an isomorphism. (Exercise)

In fact, we can say something much more general once we get to deformation retracts and homotopy equivalence.

Retractions

Def: Let $A \subseteq X$, then if $r : X \to A$ is a continuous map s.t.
$r(a) = a \ \forall\, a \in A$, r is called a retraction of X onto A,



and A is a retract of X.

Then if $i : (A, a_0) \to (X, a_0)$ is the inclusion, we have

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[rr, bend right, "id"'] & X \arrow[r, "r"] & A \end{tikzcd}$$ so $r_* \circ i_* = id \Rightarrow i_*$ is injective and r_* is surjective.